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Dielectric response of composites with graded cylindrical particles

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Abstract
Effective dielectric responses of graded cylindrical composites are investigated when an external uniform field is applied to the composites. Considering linear random composites of cylindrical particles with a specific dielectric function, which varies along the radial direction of the particles, we have studied three cases of dielectric profiles: exponential function, linear and power-law profiles. For each case, the effective dielectric response of graded composites is given on the basis of exact solutions of the local potentials of composites in the dilute limit. For a larger volume fraction, we have proposed an effective medium approximation to estimate the effective dielectric response.

1. Introduction

In nature, graded materials, whose properties are graded in space along one direction, are abundant, for example, bamboo and cells in living systems [1, 2]. The physical properties of graded materials can be designed for specific needs in engineering by changing their composition or microstructure [3–6]. For instance, the thermal conductivity, dielectric constant and electric conductivity can be designed to vary along the radius in a cylindrical or spherical particle [7, 8]. Recently, Dong *et al* [9] have developed a first-principles approach to compute the effective response of graded spherical composites. If the composites contain inclusions of the graded materials, the effective properties of the graded composite media are very different from those of the homogeneous medium because of the effect of the gradient function on the effective response. The formulae of homogeneous composites are not applied to estimate the response of graded composites [10]. Therefore it is necessary to develop new methods to investigate the effective response of graded composites under an external applied field.

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In this paper, for the linear constitutive relationship between electric displacement D and electric field E, $D = \varepsilon E$, we have investigated the effective dielectric response of graded cylindrical composites under an external uniform applied electric field. Considering three dielectric functions of cylindrical inclusion, $\varepsilon_i(r) = \mathrm{e}^{\beta r}$, $b + cr$ and $c_k r^k$ (where, r is the radial variable of cylindrical inclusion in cylindrical coordinates and β, b, c, c_k and k are constants), we have derived the effective response of the graded cylindrical composites by the effective medium approximation for each case of the dielectric functions of cylindrical inclusions.

Under a quasi-electrostatic approach, we assume that the corresponding governing equations are $\nabla \cdot D = 0$ and $\nabla \times E = 0$. The boundary conditions of the composites are the continuity of the local potentials of the inclusion particle and the normal components of the electric displacement at the surface of the inclusion particle. At infinity the potential must match that of the external applied field.

2. Potentials in graded composites

Let us consider a cylindrical particle with dielectric function, $\varepsilon_i(r)$, embedded in a homogeneous and isotropic host with dielectric constant ε_m. The cylindrical particle has unit radius. If the external electric field $E_a = E_0\hat{x}$ is applied to the composites along the $\hat{x}$ direction, in cylindrical coordinates, the differential equations for the potential in the whole region can be reduced to two dimensions:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\varepsilon_\alpha(r) r \frac{\partial \Phi_\alpha(r,\phi)}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[\varepsilon_\alpha(r)\frac{1}{r}\frac{\partial \Phi_\alpha(r,\phi)}{\partial \phi}\right] = 0, \qquad \text{in } \Omega_\alpha,\ \alpha = i, m \tag{1}$$

where the subscripts $\alpha = i, m$ denote the quantities in the inclusion (i) and host (m) regions, respectively. $\Phi_i(r, \varphi)$ and $\Phi_m(r, \varphi)$ denote the potentials of the inclusion region and host region, respectively. Ω_α is the region of the α type material. In the host region, the potential $\Phi_m(r, \phi)$ can be derived easily using equation (1) and the boundary condition at infinity:

$$\Phi_m(r, \theta) = -(r + Br^{-1})E_0 \cos(\phi).$$

For the cylindrical inclusion, using the method of separation of variables, we can express the general solution of the potential $\Phi_i(r, \phi)$ in the form

$$\Phi_i(r, \theta) = \sum_{n=0}^{\infty} R_n(r)\cos(n\phi). \tag{2}$$

The equation of the radial part, $R_n(r)$, of the potential is governed by equation (3):

$$\frac{1}{\varepsilon_i(r)}\frac{\mathrm{d}\varepsilon_i(r)}{\mathrm{d}r}\frac{\mathrm{d}R_n(r)}{\mathrm{d}r} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mathrm{d}R_n(r)}{\mathrm{d}r}\right) - \frac{n^2}{r^2}R_n(r) = 0. \tag{3}$$

Next we will derive the potential in a cylindrical particle with the dielectric function, $\varepsilon_i(r)$, for the following three cases. Case (A): exponential function $\varepsilon_i(r) = \mathrm{e}^{\beta r}$. For case (B): linear profile $\varepsilon_i(r) = b + cr$. In case (C): power-law profile $\varepsilon_i(r) = c_k r^k$.

Case (A). *For a cylindrical particle with exponential function $\varepsilon_i(r) = \mathrm{e}^{\beta r}$, equation (3) is rewritten in the form*

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_n(r)}{\mathrm{d}r}\right) + \beta\frac{\mathrm{d}R_n(r)}{\mathrm{d}r} - \frac{n^2}{r^2}R_n(r) = 0. \tag{4}$$

Here, the Frobenius method [11] is applied to solve equation (4). Taking $R_n(x) = r^p\sum_{k=0}^{\infty} a_k^n r^k$, and substituting it into equation (4), we have

$$\sum_{k=0}^{\infty}[p(p-1) + p - n^2]a_k^n(r/2)^{k+p} + \sum_{k=0}^{\infty}[2p(k+1)a_{k+1}^n + (k+1)a_{k+1}^n + 2\beta p a_k^n](r/2)^{k+p+1}$$

$$+\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}^{n}+2\beta(k+1)a_{k+1}^{n}](r/2)^{k+p+2}=0. \qquad (5)$$

The characteristic exponent p is determined if we set the coefficients $p(p-1)+p-n^2$ of the low power term r^{k+p} to be equal to zero:

$$p(p-1)+p-n^2=0. \qquad (6)$$

From equation (6), we obtain $p=n$ or $-n$. Thus equation (5) can be simplified to equation (7):

$$(2pa_1^n+a_1^n+2\beta pa_0^n)(r/2)^{p+1}+\sum_{k=0}^{\infty}[2p(k+2)a_{k+2}^{n}+(k+2)a_{k+2}^{n}+2\beta pa_{k+1}^{n}](r/2)^{k+p+2}$$
$$+\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}^{n}+2\beta(k+1)a_{k+1}^{n}](r/2)^{k+p+2}=0. \qquad (7)$$

If we set the coefficients of the power series r^{p+1} and r^{k+p+2} to be zero in equation (7), we can obtain the iteration relationships between a_k^n:

$$(2p+1)a_1^n+\beta pa_0^n=0, \qquad (8)$$
$$(2p+k+2)(k+2)a_{k+2}^{n}+\beta(p+k+1)a_{k+1}^{n}=0. \qquad (9)$$

From equations (8) and (9), we have, respectively,

$$a_1^n=-\frac{\beta p}{(2p+1)}a_0^n \qquad (10)$$

and

$$a_{k+2}^{n}=-\frac{\beta(p+k+1)}{(k+2)(2p+k+2)}a_{k+1}^{n}, \qquad k=0,1,2,\ldots. \qquad (11)$$

Considering equations (10) and (11), we get the general iteration formula

$$a_{k+1}^{n}=-\frac{\beta(p+k)}{(k+1)(2p+k+1)}a_{k}^{n}, \qquad k=0,1,2,\ldots \qquad (12)$$

where a_0^n is constant. More generally, we set $a_0^n=1$. Clearly, the series solution is convergent in the range $|r|<\infty$.

Hence the general solution of the radial part of the potential is

$$R_n(r)=A_nr^n\sum_{k=0}^{\infty}a_k^nr^k+A_{-n}r^{-n}\sum_{k=0}^{\infty}a_k^{-n}r^k.$$

Because the potential of the inclusion is finite and constant at the polar point $r=0$, we can determine the valid solution of the radial part, $R_n(r)=A_nr^n\sum_{k=0}^{\infty}a_k^nr^k$. Thus the potential of the cylinder is

$$\Phi_i(r,\theta)=\sum_{n=0}^{\infty}A_nr^n\left(\sum_{k=0}^{\infty}a_k^nr^k\right)E_0\cos(n\phi).$$

Using the boundary conditions: the continuity of the local potentials and normal components of the electric displacement at the surface of the inclusion particles, we can determine the coefficients of the potentials in the host and inclusion regions. The potential in a cylindrical particle is

$$\Phi_i(r,\theta)=A_1rE_0\cos(\phi)\sum_{k=0}^{\infty}a_k^1r^k$$

where $a_k^1 = (-\beta)^k/(k+2)!, a_0^1 = 1$. Considering $\sum_{k=0}^{\infty}(-\beta r)^{k+2}/(k+2)! = (\mathrm{e}^{-\beta r} - 1+\beta r)$, the potential of a cylindrical inclusion can be rewritten in the form

$$\Phi_i(r,\phi) = \frac{A_1\cos(\phi)E_0}{r\beta^2}(\mathrm{e}^{-\beta r} - 1 + \beta r), \qquad r \leqslant 1 \tag{13}$$

where $A_1 = -2\varepsilon_m/(v_1\varepsilon_m + v_2\mathrm{e}^{\beta})$ and

$$v_1 = (\mathrm{e}^{-\beta} + \beta - 1)/\beta^2,$$
$$v_2 = [1 - \mathrm{e}^{-\beta}(1+\beta)]/\beta^2.$$

In the host region, the potential is

$$\Phi_m(r,\theta) = -(r + Br^{-1})E_0\cos(\phi), \qquad r \geqslant 1 \tag{14}$$

where $B = (\varepsilon_m v_1 - \mathrm{e}^{\beta}v_2)/(\varepsilon_m v_1 + \mathrm{e}^{\beta}v_2)$.

Case (B). *For the linear profile, $\varepsilon_i(r) = b + cr$, equation (3) is reduced to equation (15):*

$$\frac{\mathrm{d}^2R_n(r)}{\mathrm{d}r^2} + \frac{1}{\xi + r}\frac{\mathrm{d}R_n(r)}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}R_n(r)}{\mathrm{d}r} - \frac{n^2}{r^2}R_n(r) = 0. \tag{15}$$

Taking the variable transformation, $R_n(z) = z^k u(z)$, $r = -\xi z$, $\xi = b/c$, we can simplify equation (15) to the following form:

$$z^2(z-1)\frac{\mathrm{d}^2u(z)}{\mathrm{d}z^2} + [(2k+2)z - 2k - 1]z\frac{\mathrm{d}u(z)}{\mathrm{d}z} + [(k^2 + k - n^2)z - k^2 + n^2]u(z) = 0. \tag{16}$$

In order to identify the characteristic exponent k, we set $-k^2 + n^2 = 0$ (so $k = \pm n$). Then equation (16) can be transformed into equation (17):

$$z(z-1)\frac{\mathrm{d}^2u(z)}{\mathrm{d}z^2} + [\gamma - (\alpha + \beta + 1)z]\frac{\mathrm{d}u(z)}{\mathrm{d}z} - \alpha\beta u(z) = 0, \tag{17}$$

where $\gamma = 2k+1$, $\alpha + \beta = 2k+1$, $\alpha\beta = -n^2 + k(k+1)$. The solution $u(z)$ of equation (17) is the hypergeometric function $F(\alpha,\beta,\gamma,z)$ [12]. The hypergeometric function is investigated comprehensively and it is analytic over all complex planes besides its singular points [12]. Therefore the function $R_n(r)$ can be expressed as

$$R_n(r) = A_n\left(-\frac{r}{\xi}\right)^n F\left(\alpha_n, \beta_{n,}\gamma_n, -\frac{r}{\xi}\right) + A_{-n}\left(-\frac{r}{\xi}\right)^{-n} F\left(\alpha_{-n}, \beta_{-n}, \gamma_{-n}, -\frac{r}{\xi}\right), \tag{18}$$

where $\gamma_{\pm n} = \pm 2n + 1$, $\alpha_{\pm n} = [(1 \pm 2n) \mp \sqrt{1+4n^2}]/2$ and $\beta_{\pm n} = [(1 \pm 2n) \pm \sqrt{1+4n^2}]/2$.

Using the boundary condition on the surface of a cylindrical inclusion and a finite constant of the potential $\Phi_i(r,\theta)$ at the polar point $r = 0$, we determine the coefficients of the potentials in the host and inclusion regions. So the potentials are

$$\Phi_m(r,\theta) = -(r + Dr^{-1})E_0\cos(\phi), \qquad r \geqslant 1,$$
$$\Phi_i(r,\theta) = -A_1\frac{r}{\xi}F\left(\alpha_1, \beta_1, \gamma_1, -\frac{r}{\xi}\right)E_0\cos(\phi), \qquad r < 1,$$

where

$$A_1 = -2\varepsilon_m/[\varepsilon_m v_1 + (b+c)v_2],$$
$$D = [\varepsilon_m v_1 - (b+c)v_2]/[\varepsilon_m v_1 + (b+c)v_2],$$
$$v_1 = \xi' F(\alpha_1, \beta_{1,}\gamma_1, \xi'),$$
$$v_2 = \xi'\left[F(\alpha_1, \beta_1, \gamma_1, \xi') + \xi'\frac{\alpha_1\beta_1}{\gamma_1}F(\alpha_1 + 1, \beta_1 + 1, \gamma_1 + 1, \xi')\right],$$

$$\xi' = -c/b,$$
$$\alpha_1 = (3 - \sqrt{5})/2,$$
$$\beta_1 = (3 + \sqrt{5})/2,$$
$$\gamma_1 = 3.$$

Case (C). *For the power-law profile, $\varepsilon_i(r) = c_k r^k$, Gu and Yu [13] have given the exact solutions of potentials in the cylindrical particle and host regions:*

$$\Phi_m(r,\theta) = (r + D_1 r^{-1}) E_0 \cos(\phi), \qquad r \geqslant 1$$
$$\Phi_i(r,\theta) = A_1 r^{s_k} E_0 \cos(\phi), \qquad r < 1$$

where

$$s_k = (\sqrt{k^2+4} - k)/2,$$
$$D_1 = (\varepsilon_m - c_k s_k)/(\varepsilon_m + c_k s_k),$$
$$A_1 = 2\varepsilon_m/(\varepsilon_m + c_k s_k).$$

3. Effective dielectric response

On the basis of the potential in the inclusion region, we can calculate the effective dielectric response of the graded composites:

$$\frac{1}{V}\int_V [D - \varepsilon_m E]\,\mathrm{d}v = \bar{D} - \varepsilon_m \bar{E}, \tag{19}$$

where V is the volume of the composite media. $\bar{D}$ and $\bar{E}$ are the volume averages of the electric displacement and electric field over the whole composite regions, respectively. We define the effective dielectric response ε_e as $\bar{D} = \varepsilon_\mathrm{e}\bar{E}$. Using the relationship $D = \varepsilon_m E$ in the host region and substituting it into the left-hand side of equation (19), we can rewrite equation (19) in the form:

$$\frac{1}{V}\int_{\Omega_i} [\varepsilon_i(r) - \varepsilon_m] E\,\mathrm{d}v = (\varepsilon_\mathrm{e} - \varepsilon_m)\bar{E}. \tag{20}$$

In the dilute limit, we can evaluate the effective response of the composite by solving the electric field of a single particle under an external uniform electric field $\bar{E}$. In this case, the effective dielectric response ε_e can be estimated from equation (20) for the three cases of dielectric functions by substituting the potentials of the inclusion region into the left-hand side of equation (20).

Case (A). *For the exponential function $\varepsilon_i(r) = \mathrm{e}^{\beta r}$, in the dilute limit, the effective dielectric response is*

$$\varepsilon_\mathrm{e}/\varepsilon_m = 1 + 2 f_i (v_3 - \varepsilon_m v_4)/(\varepsilon_m v_1 + v_2 \mathrm{e}^{\beta}), \tag{21}$$

where f_i is the volume fraction of the cylindrical inclusion:

$$v_3 = (\mathrm{e}^{\beta} - \beta - 1)/\beta^2,$$
$$v_4 = (\mathrm{e}^{-\beta} + \beta - 1)/\beta^2.$$

Now we demonstrate that, in the dilute limit, equation (21) can be reduced to the classical formulae by letting $\beta \to 0$. Taking the following limits, $\lim_{\beta\to 0} v_3/v_1 = 1$, $\lim_{\beta\to 0} v_4/v_1 = 1$

and $\lim_{\beta\to 0} v_2/v_1 = 1$, we get Maxwell's formula of the cylindrical particle with the dielectric constant $\varepsilon_i = 1$:

$$\varepsilon_e/\varepsilon_m = 1 + 2\lim_{\beta\to 0} f_i(v_3 - \varepsilon_m v_4)/(\varepsilon_m v_1 + v_2 e^{\beta}) = 1 + 2f_i(1-\varepsilon_m)/(1+\varepsilon_m).$$

Case (B). *For the linear profile, $\varepsilon_i(r) = b + cr$, the effective dielectric response is*

$$\varepsilon_e/\varepsilon_m = 1 + 2f_i[(b-\varepsilon_m)v_3 + cv_4]/[\varepsilon_m v_1 + (b+c)v_2], \tag{22}$$

where f_i is the volume fraction of the cylindrical inclusion:

$$\begin{aligned}
v_3 = {} & 2\xi'[w(1,\alpha_1,\beta_1,\gamma_1,\xi') - \tilde{w}(2,\alpha_1,\beta_1,\gamma_1,\xi')] \\
& + \frac{\alpha_1\beta_1}{\gamma_1}(\xi')^2[w(1,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi') - 2w(2,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi') \\
& + 2\tilde{w}(3,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi')] \\
v_4 = {} & 2\xi'[w(1,\alpha_1,\beta_1,\gamma_1,\xi') - 2w(2,\alpha_1,\beta_1,\gamma_1,\xi') + 2\tilde{w}(2,\alpha_1,\beta_1,\gamma_1,\xi')] \\
& + \frac{\alpha_1\beta_1}{\gamma_1}(\xi')^2[w(1,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi') - 3w(2,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi') \\
& + 6w(3,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi') - 6\tilde{w}(4,\alpha_1+1,\beta_1+1,\gamma_1+1,\xi')] \\
\tilde{w}(n,\alpha,\beta,\gamma,\xi') = {} & w(n,\alpha,\beta,\gamma,\xi') - w(n,\alpha,\beta,\gamma,0), \\
w(n,\alpha,\beta,\gamma,z) = {} & (\xi')^{-n}F(\alpha-n,\beta-n,\gamma-n,z)\prod_{i=1}^{n}\frac{(\gamma-i)}{(\alpha-i)(\beta-i)}.
\end{aligned}$$

For the case $c \to 0$, the classical Maxwell's formula of homogeneous composite media will be obtained for $\varepsilon_i = b$ from equation (22). In this case, we easily obtain the following limits referencing the hypergeometric function: $\lim_{c\to 0} v_2/v_1 = 1$, $\lim_{c\to 0} v_3/v_1 = 1$ and $\lim_{c\to 0} v_4/v_1 = 2/3$. Substituting the three limits into equation (23), we have

$$\varepsilon_e/\varepsilon_m = 1 + \lim_{c\to 0}\frac{2f_i[(b-\varepsilon_m)\frac{v_3}{v_1} + c\frac{v_4}{v_1}]}{\varepsilon_m + (b+c)\frac{v_2}{v_1}} = 1 + 2f_i(b-\varepsilon_m)/(\varepsilon_m + b).$$

This result is the classical formula of Maxwell's effective response of cylindrical composites for $\varepsilon_i = b$.

Case (C). *For the power-law profile, $\varepsilon_i(r) = c_k r^k$, we have*

$$\varepsilon_e/\varepsilon_m = 1 + 2f_i(s_k+1)(c_k I_2 - \varepsilon_m I_1)/(\varepsilon_m + c_k s_k), \tag{23}$$

where f_i is the volume fraction of the cylindrical inclusion:

$$I_1 = (s_k+1)^{-1},$$
$$I_2 = (s_k+k+1)^{-1}.$$

For $k \to 0$, the classical Maxwell's formula is again obtained from equation (23). Letting $k \to 0$, we have the limits $\lim_{k\to 0} s_k = 1$, $\lim_{k\to 0} I_1 = 1/2$ and $\lim_{k\to 0} I_2 = 1/2$. With these limits, we have Maxwell's formula of cylindrical composites with the dielectric constant $\varepsilon_i = c_0$:

$$\varepsilon_e/\varepsilon_m = 1 + \lim_{k\to 0} 2f_i(s_k+1)(c_k I_2 - \varepsilon_m I_1)/(\varepsilon_m + c_k s_k) = 1 + 2f_i(c_0 - \varepsilon_m)/(c_0 + \varepsilon_m).$$

Now, we will give an effective medium approximation (EMA) to estimate the effective dielectric response for a larger volume fraction. In order to estimate the average field $\bar{E}$ over the whole composite media regions, we consider a sample of cylindrical particles with

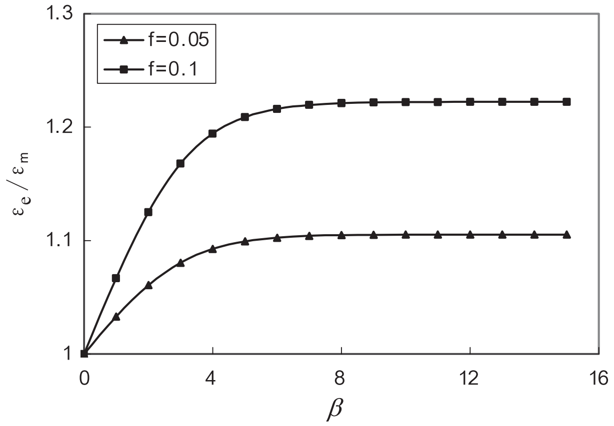

Figure 1. For a cylindrical inclusion with an exponential dielectric function, $\varepsilon_i(r) = \mathrm{e}^{\beta r}$, the effective response contrast $\varepsilon_\mathrm{e}/\varepsilon_m$ is plotted against the parameter β for a volume fraction $f = 0.05$ and 0.1.

the unknown effective dielectric response ε_e and embed it in a host medium with dielectric constant ε_m. The external field $E_a = E_0\hat{x}$ is applied to the new composite media along the $\hat{x}$ direction. Thus, as an EMA, we can regard the average field over the sample cylindrical region as that of the whole original composite media region. In this case, we can get [14]

$$\bar{E} = 2\varepsilon_m E_a/(\varepsilon_\mathrm{e} + \varepsilon_m). \tag{24}$$

Substituting equation (24) into the right-hand side of equation (20), we obtain new effective dielectric responses for the three cases of cylindrical composites. The formulae are approximately suitable for a larger volume fraction compared with the dilute limit formulae (21)–(23). Here we should note that our EMA method does not consider exactly the effects of the neighbouring inclusions on the local electric field around each particle. In the present case, our formulae are only valid for low concentrations and the concentration limit is less than 0.1. For high concentrations, Tuncer and others have investigated the effects of neighbouring particles on the effective dielectric response [15, 16].

Case (A). *In the case of the exponential function $\varepsilon_i(r) = \mathrm{e}^{\beta r}$, for a larger volume fraction, we have*

$$\varepsilon_\mathrm{e}/\varepsilon_m = 1 + 2f_i A_1(\varepsilon_m v_4 - v_3)/[2\varepsilon_m - f_i A_1(\varepsilon_m v_4 - v_3)]. \tag{25}$$

Case (B). *For the linear profile, $\varepsilon_i(r) = b + cr$, we have*

$$\varepsilon_\mathrm{e}/\varepsilon_m = 1 + 2f_i[(b - \varepsilon_m)v_3 + cv_4]/[\varepsilon_m v_1 + (b + c)v_2 - f_i((b - \varepsilon_m)v_3 + cv_4)]. \tag{26}$$

Case (C). *For the power-law profile, $\varepsilon_i(r) = c_k r^k$, at a larger volume fraction, we have*

$$\varepsilon_\mathrm{e}/\varepsilon_m = 1 + f_i(s_k + 1)(c_k I_2 - \varepsilon_m I_1)/[\varepsilon_m + c_k s_k - f_i(s_k + 1)(c_k I_2 - \varepsilon_m I_1)]. \tag{27}$$

In discussion, we should note that, for lower concentrations of inclusions, equations (25)–(27) can be exactly reduced to the dilute limit formulae (21)–(23), respectively, if we remain at the first power of the volume fraction f_i. It is instructive to show the effects of the gradient parameters of dielectric functions on the effective responses of composites by means of equations (25)–(27). In figure 1, we plot the effective dielectric response of cylindrical

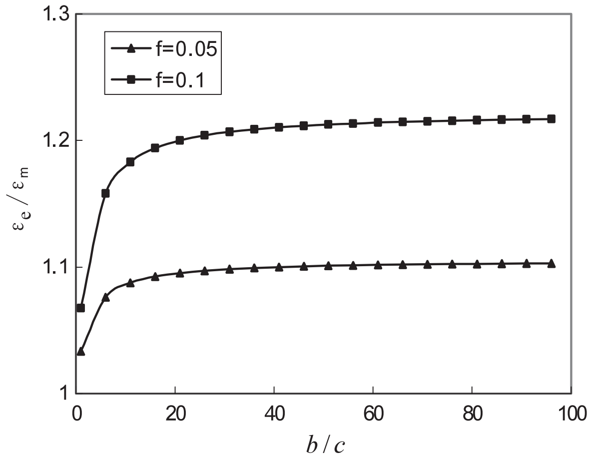

Figure 2. For the linear dielectric profile $\varepsilon_i(r) = b + cr$ of a cylindrical inclusion, the effective response $\varepsilon_e/\varepsilon_m$ is plotted against the contrast parameter b/c at a volume fraction $f = 0.05$ and 0.1.

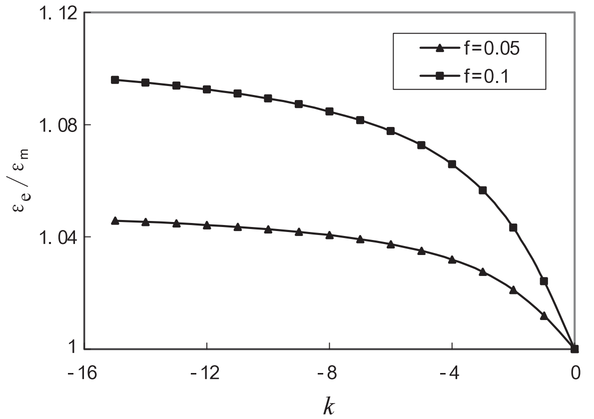

Figure 3. For the power-law profile $\varepsilon_i(r) = c_k r^k$ of a cylindrical inclusion, at a volume fraction $f = 0.05$ and 0.1, the effective response contrast $\varepsilon_e/\varepsilon_m$ is plotted against the parameter k for $c_k = 1$.

composites with exponential dielectric profile $\varepsilon_i(r) = \mathrm{e}^{\beta r}$ versus the parameter β. Clearly, the effective response $\varepsilon_e/\varepsilon_m$ increases as the parameter β increases. For the linear dielectric function $\varepsilon_i(r) = b + cr$, a similar conclusion is obtained from figure 2. The effective response contrast $\varepsilon_e/\varepsilon_m$ increases as the ratio b/c increases. For both dielectric function profiles, it is attributed to the fact that the dielectric constants of inclusions increase when the parameters β and b/c increase, respectively. In figure 3, the effective response is plotted against the power-law parameter k. In contrast to parameters β and b/c, the effective response decreases as the parameter k increases.

4. Conclusions

The effective dielectric response of linear random composites with graded cylindrical material is investigated. We have exactly derived the local potentials of cylindrical composites for the exponential and linear dielectric functions. On the basis of the potentials in the inclusion region, in the dilute limit, the formulae of the effective dielectric responses are derived for three cases of the exponential, linear and power-law dielectric functions. Furthermore, for a larger volume fraction of inclusion, an EMA is given to estimate the effective response. In fact, for spherical composites with exponential, linear and power-law dielectric functions changing along its radial direction, our method can be used to deal with them. In addition, using our results for the three graded cylindrical composites, we can extend them to treat the effective response of Kerr-like weakly non-linear composites if the linear and non-linear dielectric responses of inclusions are functions (exponential function, linear and power-law profile in this paper) and constants, respectively, by means of the perturbation approach [17]. For other complicated gradient functions of the dielectric constant of the inclusion, one should use numerical methods, such as the finite element method [16, 18] or boundary-integral equations [19], to tackle it so that the effective response of the graded composites can be used to improve engineering designs and applications [20].

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References

[1] Amada S 1995 *MRS Bull.* **20** 35
[2] Holt J B, Koizumi M, Hirai T and Munir Z A 1993 *Ceramic Transaction: Functionally Graded Material* vol 34 (Westerville, OH: The American Ceramic Society)
[3] Watanabe R 1995 *MRS Bull.* **20** 32
[4] Rabin B H, Williamson and Suresh S 1995 *MRS Bull.* **20** 37
[5] Erdogan F, Kaya A C and Joseph P E 1991 *J. Appl. Mech.* **58** 410
[6] Parameswaran V and Shukla A 2000 *J. Mater. Sci.* **35** 21
[7] Park C W 2000 *Indust. Eng. Chem. Res.* **39** 79
[8] Manhart P K and Blankenbecler R 1997 *Opt. Eng.* **36** 1607
[9] Dong L, Gu G Q and Yu K W 2003 *Phys. Rev.* B **67** 224205
[10] Maxwell J C 1873 *Electricity and Magnetism* 1st edn (New York: Oxford University Press) p 365
[11] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Sciences and Engineers* (New York: McGraw-Hill)
[12] Bailey W N (ed) 1935 *Generalized Hypergeometric Series* (Cambridge: Cambridge University Press)
[13] Gu G Q and Yu K W 2003 *J. Appl. Phys.* **94** 3376
[14] Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Medium* vol 8 (Oxford: Pergamon)
[15] Tuncer E, Serdyuk Y V and Gubanski S M 2002 *IEEE Trans. Dielectr. Electr. Insul.* **9** 809
[16] Tuncer E, Nettelblad B and Gubanski S M 2002 *J. Appl. Phys.* **92** 4612
[17] Gu G Q and Yu K W 1992 *Phys. Rev.* B **46** 4502
[18] Tuncer E, Gubanski S M and Nettelblad B 2001 *J. Appl. Phys.* **89** 8092
[19] Brosseau C and Beroual A 2003 *Prog. Mater. Sci.* **48** 373
[20] Bishop A *et al* 1993 *J. Mater. Sci. Lett.* **12** 1516